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ON GENERALIZED HANKEL OPERATORS INDUCED BY SUMS AND PRODUCTS†

BY

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ABSTRACT

Vectorial Hankel operators are studied, in particular the ranges of Hankel operators induced by sums and products of matrix functions defined on the unit circle are determined. The analytical tools involve factorization theorems for operator valued analytic functions and the spectral analysis of operators that intertwine restricted shifts.

1. Introduction

In this paper we continue our investigation of vectorial Hankel operators started in [5], [6], [7]. We will study in depth the ranges of Hankel operators induced by sums and products of matrix functions defined on the unit circle.

The results obtained here have a system theoretic interpretation and are closely related to questions of controllability and observability of series and parallel connection of canonical linear systems. For a preliminary exposition of these ideas the reader is referred to [8].

Whereas some of the results of this paper may be stated in somewhat greater generality, we will restrict ourselves to Hankel operators induced by matrix valued functions. The basic analytical tools will be the factorization theorems obtained in [6], [7] and the spectral analysis of operators that intertwine restricted shifts [2], [3].

Let M and N be two separable complex Hilbert spaces. We denote by $L^2(N)$ the Hilbert space of all (equivalence classes) weakly measurable functions from the unit circle to N having finite norm. The norm in $L^2(N)$ is the one induced by the inner product

$$(f,g) = \frac{1}{2\pi}\int_0^{2\pi} (f(e^{it}), g(e^{it}))_N dt.$$

We let $H^2(N)$ denote the subspace of $L^2(N)$ of all functions whose negative indexed Fourier coefficients are all zero. We recall that $H^2(N)$ functions have analytic extensions into the unit disc from which they can be recaptured as radial limits almost everywhere [9]. We will always use the same letter for both the $H^2(N)$ function as defined on the unit circle, and its analytic extension to

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the open unit disc. We let χ denote the identity function on the closed unit disc, i.e. $\chi(z) = z$. We define right shift S in $H^2(N)$ by letting $Sf = \chi f$ for all f in $H^2(N)$. We note that its adjoint S^* is given by $(S^*f)(z) = (f(z) - f(0))/z$. We let $B(N,M)$ be the space of all bounded linear operators from N to M equipped with the operator norm. By $L^\infty(B(N,M))$ we denote the space of all (equivalence classes) weakly measurable essentially bounded functions from the unit circle to $B(N,M)$. Elements of $L^\infty(B(N,M))$ have Fourier expansions [9], and we will denote by $H^\infty(B(N,M))$ the subspace of $L^\infty(B(N,M))$ of all functions whose negative indexed Fourier coefficients are all zero. Functions in $H^\infty(B(N,M))$ have analytic extensions into the open unit disc and can be recaptured as strong radial limits almost everywhere. For A in $L^\infty(B(N,M))$ we let $\tilde{A}(z) = A(\bar{z})^*$. Clearly $\tilde{A}$ is in $L^\infty(B(M,N))$.

A subspace of $H^2(N)$ will be called right invariant or left invariant if it is invariant under the right or left shift, respectively. The orthogonal complement of a left invariant subspace is right invariant, and vice versa. A subspace K of $H^2(N)$ is right invariant if and only if it has a representation $K = PH^2(N)$ for some function P in $H^\infty(B(N,N))$ which has norm bounded by one and almost everywhere $P(e^{it})$ is a partial isometry with a fixed initial space [9]. Such functions are called rigid. An important subclass of rigid functions are the inner functions, i.e., those for which almost everywhere on the unit circle $P(e^{it})$ is unitary. Right invariant subspaces that correspond to inner functions are called subspaces of full range [9]. Given an inner function P in $H^\infty(B(N,N))$, we denote by $H(P)$ the left invariant subspace $\{PH^2(N)\}^\perp$, where the orthogonal complement is taken in $H^2(N)$. We will use $P_{H(P)}$ for the orthogonal projection of $H^2(N)$, and sometimes of $L^2(N)$, onto $H(P)$. We define an operator $S(P)$ in $H(P)$ by

$$S(P)f = P_{H(P)}\chi f \tag{1.1}$$

for all f in $H(P)$. $S(P)$ is called the restricted right shift and we have $S(P)^* = S^*|H(P)$, that is $S(P)^*$ is the restriction of the left shift to the left invariant subspace $H(P)$.

An inner function P in $H^\infty(B(M,M))$ is a left inner factor of a function A in $H^\infty(B(N,M))$ if $A = PA'$ for some A' in $H^\infty(B(N,M))$. Two functions A and A_1 in $H^\infty(B(N,M))$ and $H^\infty(B(N_1,M))$ respectively are left prime if A and A_1 have no common nontrivial left inner factor. We will use the notation $(A,A_1)_L = I_M$ to denote the left primeness of A and A_1. We will say that A and A_1 are strongly left prime if there exists a $\delta > 0$ such that for all z in the open unit disc

$$\inf\{\|A(z)^*\xi\| + \|A_1(z)^*\xi\| \mid \xi \in M,\ \|\xi\| = 1\} \geqq \delta. \tag{1.2}$$

We will use $[A, A_1]_L = I_M$ to denote the strong left primeness of A and A_1. Similarly, given A in $H^\infty(B(N,M))$ and A_1 in $H^\infty(B(N,M_1))$, we define right and strong right primeness analogously. We clearly have $(A,A_1)_R = I_N$ if and only if $(\tilde{A},\tilde{A}_1)_L = I_N$, and $[A,A_1]_R = I_N$ if and only if $[\tilde{A},\tilde{A}_1]_L = I_N$. Thus $[A,A_1]_R = I_N$ is equivalent to the existence of a $\delta > 0$ such that for all z in the open unit disc

$$\inf\{\|A(z)\xi\| + \|A_1(z)\xi\| \mid \xi \in N,\ \|\xi\| = 1\} \geqq \delta. \tag{1.3}$$

Let J be the unitary map in $L^2(N)$ defined by $(Jf)(e^{it}) = f(e^{-it})$. Given A in $L^\infty(B(N,M))$, we define H_A the Hankel operator induced by A as the operator from $H^2(N)$ into $H^2(M)$ defined by

$$H_A f = P_{H^2(M)} A(Jf) \tag{1.4}$$

for all f in $H^2(N)$. It is easy to check that

$$S^* H_A = H_A S. \tag{1.5}$$

Here we used the same notation for the shift operators in $H^2(N)$ and $H^2(M)$. From (1.5) it follows that $\ker H_A$ is a right invariant subspace of $H^2(N)$ whereas $\overline{\text{Range } H_A}$ is a left invariant subspace of $H^2(M)$. We will say that A is strictly noncyclic if $\{\text{Range } H_A\}^\perp$ is an invariant subspace of full range. Here the orthogonal complement is taken in $H^2(M)$. For a strictly noncyclic function A the two invariant subspaces introduced above are closely associated with factorizations of the function A on the unit circle. The following theorem is quoted from [7].

THEOREM 1.1. *Let A be in $L^\infty(B(N,M))$.*

(a) *The following three statements are equivalent:*

(i) *A is strictly noncyclic.*

(ii) *A is a strong radial of a $B(N,M)$ valued meromorphic function of bounded type in $D_e = \{z \mid 1 < |z| \leqq \infty\}$.*

(iii) *On the unit circle A has a factorization*

$$A = \bar{\chi} P C^*, \tag{1.6}$$

where P is inner in $H^\infty(B(M,M))$ and C is in $H^\infty(B(M,N))$ and

$$(P,C)_R = I_M. \tag{1.7}$$

(iv) *On the unit circle A has a factorization*

$$A = \tilde{\chi} C_1^* P_1, \tag{1.8}$$

where P_1 *is inner in* $H^\infty(B(N,N))$ *and* C_1 *is in* $H^\infty(B(M,N))$ *and*

$$(P_1, C_1)_L = I_N. \tag{1.9}$$

(b) *If N and M are finite dimensional then the inner functions P and* P_1 *are quasiequivalent* [10], [11] *and in particular* $\det P = \det P_1$ *is satisfied.*

We will refer to the factorizations (1.6) and (1.8) satisfying (1.7) and (1.9) as right and left prime factorizations, respectively. The inner functions P and P_1 are associated with the range and kernel of the Hankel operator H_A by the following relations:

$$\{\text{Range } H_A\}^\perp = PH^2(M) \tag{1.10}$$

and

$$\ker H_A = \tilde{P}_1 H^2(N). \tag{1.11}$$

A slight generalization of the results of [5] yield the following theorem about range closure of Hankel operators.

THEOREM 1.2. *Let* A *in* $L^\infty(B(N,M))$ *be strictly noncyclic and have the prime factorizations* (1.6) *and* (1.8).

(i) Range $H_A = H(P)$ *if and only if* $[P,C]_R = I_M$.

(ii) Range $H_A = H(P)$ *if and only if* $[P_1, C_1]_L = I_N$.

2. Hankel operators induced by products

Let L, M and N be three finite dimensional Hilbert spaces, and let A and B be strictly noncyclic functions in $L^\infty(B(N,M))$ and $L^\infty(B(L,N))$, respectively. By Theorem 1.1, the functions A and B have the following factorizations on the unit circle:

$$A = \tilde{\chi} P C^* = \tilde{\chi} C_1^* P_1 \tag{2.1}$$

and

$$B = \tilde{\chi} R D^* = \tilde{\chi} D_1^* R_1, \tag{2.2}$$

where $P \in H^\infty(B(M,M))$, P_1, $R \in H^\infty(B(N,N))$ and $R_1 \in H^\infty(B(L,L))$ are inner functions and C, $C_1 \in H^\infty(B(M,N))$ and D, $D_1 \in H^\infty(B(N,L))$. Moreover we assume that

$$(P,C)_R = I_M, \quad (P_1, C_1)_L = I_N \tag{2.3}$$

and

$$(R, D)_R = I_N, \quad (R_1, D_1)_L = I_L \tag{2.4}$$

are satisfied.

Rather than study the range of the Hankel operator induced by the function AB we will study that of the Hankel operator induced by Γ in $L^\infty(B(L, M))$ defined by

$$\Gamma = \chi AB. \tag{2.5}$$

Since A and B are strictly noncyclic, both have meromorphic extensions of bounded type to D_e, and hence also Γ has such an extension. Thus Γ itself is strictly noncyclic and by Theorem 1.1 has the following factorizations on the unit circle:

$$\Gamma = \tilde{\chi} Q H^* = \tilde{\chi} H_1^* Q_1, \tag{2.6}$$

where $Q \in H^\infty(B(M, M))$ and $Q_1 \in H^\infty(B(L, L))$ are inner functions, H, $H_1 \in H^\infty(B(M, L))$ and the primeness conditions

$$(Q, H)_R = I_M, \qquad (Q_1, H_1)_L = I_L \tag{2.7}$$

are satisfied.

The analysis of the general case will be based on the two special cases $B = \tilde{\chi} R$ and $B = \tilde{\chi} D^*$.

LEMMA 2.1. *Let* $A \in L^\infty(B(N, M))$ *be strictly noncyclic and let* $R \in H^\infty(B(N, N))$ *be an inner function, then* Range $H_A \subset$ Range H_{AR}.

PROOF. Let $f \in H^2(N)$, then $\tilde{R} f \in H^2(N)$ and

$$H_{AR}(\tilde{R} f) = P_{H^2(M)} ARJ(\tilde{R} f) = P_{H^2(M)} ARR^*(Jf) = P_{H^2(M)} A(Jf) = H_A f.$$

Hence the stated range inclusion holds.

In this case $\Gamma = AR$, and from the prime factorizations (2.6) it follows that $\overline{\text{Range } H_{AR}} = H(Q)$.

LEMMA 2.2. *Let* A *and* AR *have the factorizations* (2.1) *and* (2.6) *satisfying* (2.3) *and* (2.7), *respectively, then*

$$\det Q = (\det P) \cdot (\det R) \tag{2.8}$$

if and only if

$$(R, C)_L = I_N. \tag{2.9}$$

PROOF. Assume R and C have a nontrivial greatest left inner factor S. Thus $R = SR_2$ and $C = SC_2$. Since S is nontrivial, $\det R_2 | \det R$ and $\det R \neq \det R_2$. Hence $AR = \bar{\chi}PC^*R = \bar{\chi}PC_2^*R_2$ and $(R_2, C_2)_L = I_N$. By Theorem 1.1 there exist R_3 and C_3 satisfying $(R_3, C_3) = I_M$ and for which $\bar{\chi}C_2^*R_2 = \bar{\chi}R_3C_3^*$. Since R_2 and R_3 are quasiequivalent, we have $\det R_2 = \det R_3$. Since $AR = \bar{\chi}QH^* = \bar{\chi}PR_3C_3^*$, it follows from (1.6) that $\overline{\text{Range } H_{AR}} = H(Q) \subset H(PR_3)$. The last inclusion implies $QH^2(M) \supset PR_3H^2(M)$ and hence, by Theorem 10 in [9], there exists an inner function T for which we have $QT = PR_3$. This in turn implies that $\det Q \,|\, (\det P)\cdot(\det R_3)$ and thus the equality (2.8) is impossible.

Conversely, assume (2.9) holds. We have $AR = \bar{\chi}PC^*R = \bar{\chi}QH^*$. Now, again by Theorem 1.1, C^*R has a factorization $R_2C_2^*$ satisfying $(R_2, C_2)_R = I_M$. By the quasiequivalence of R and R_2 we have the equality $\det R = \det R_2$. From Lemma 2.1, $H(P) \subset H(Q)$ and hence, by Theorem 10 in [9], there exists an inner function S in $H^\infty(B(M,M))$ for which $Q = PS$. Thus $\bar{\chi}PC^*R = \bar{\chi}QH^* = \bar{\chi}PSH^*$ and $(S,H)_R = I_M$ is implied by $(Q,H)_R = I_M$. It follows that $R_2C_2^* = SH^*$, and as both factorizations are prime we have, by (1.10), the equality $H(S) = H(R_2)$. Therefore S and R_2 differ at most by a constant unitary factor on the right. In particular $\det S = \det R_2 = \det R$ (up to a constant factor of modulus one) and thus (2.8) is satisfied.

The last lemma can be extended to yield results about the range closure of H_{AR}.

LEMMA 2.3. *Let A and AR have the factorizations* (2.1) *and* (2.6) *satisfying* (2.3) *and* (2.7), *respectively. If H_A has closed range then H_{AR} has closed range $H(Q)$ with* (2.8) *satisfied if and only if*

$$[R, C]_L = I_M. \tag{2.10}$$

PROOF. We saw in Lemma 2.2 that for (2.8) to hold (2.9) is necessary. Thus $C^*R = R_2C_2^*$ with $(R_2, C_2)_R = I_M$ and hence $AR = \bar{\chi}PC^*R = \bar{\chi}PR_2C_2^*$. From Lemma 2.2 it follows that the last factorization of AR is prime, that is $(PR_2, C_2)_R = I_M$. Thus for Range H_{AR} to be closed, it is necessary, by Theorem 1.2, that $[PR_2, C_2]_R = I_M$. Thus the weaker condition $[R_2, C_2]_R = I_M$ is also necessary and, by Theorem 1.2, this is equivalent to (2.10).

Conversely, assume (2.10) holds. This implies that the weaker condition (2.9) holds and hence by Lemma 2.1 Range H_{AR} is dense in $H(Q) = H(PR_2)$. By Lemma 3.1 in [1], which generalizes in a straightforward way to the vector valued case, we have

(2.11) $$H(PR_2) = H(P) \oplus PH(R_2).$$

Since Range $H_A = H(P) \subset$ Range H_{AR}, it suffices to show that $PH(R_2) \subset$ Range H_{AR}. Let $f \in H^2(N)$, then

$$H_{AR}f = P_{H^2(M)}AR(Jf) = P_{H^2(M)}\bar{\chi}PR_2C_2^*(Jf),$$

and hence

$$P_{PH(R_2)}H_{AR}f = P \cdot P_{H^2(M)}\bar{\chi}R_2C_2^*(Jf) = PH_{\bar{\chi}R_2C_2^*}f.$$

Since (2.10) is equivalent, by Theorem 1.2, to $[R_2, C_2]_L = I_M$, it follows that Range $H_{\bar{\chi}R_2C_2^*} = H(R_2)$, and the proof is complete.

Next we assume $B = \bar{\chi}D^*$ for D in $H^\infty(B(N,L))$ or equivalently $\Gamma = AD^*$.

Lemma 2.4. *Let A be in $L^\infty(B(N,M))$ and D in $H^\infty(B(N,L))$, then* Range $H_{AD^*} \subset$ Range H_A.

Proof. Let $f \in H^2(L)$, then

$$H_{AD^*}f = P_{H^2(M)}AD^*(Jf) = P_{H^2(M)}AJ(\tilde{D}f) = H_A(\tilde{D}f),$$

and obviously $\tilde{D}f$ is in $H^2(N)$.

Thus the factorization (2.6) of AD^* together with the primeness condition $(Q,H)_R = I_M$ imply $\overline{\text{Range } H_{AD^*}} = H(Q)$ and hence $H(Q) \subset H(P)$. By Theorem 10 in [9] it follows that for some inner function S in $H^\infty(B(M,M))$ we have $P = QS$.

Lemma 2.5. *Let A be strictly noncyclic in $L^\infty(B(N,M))$ and let A and AD^* have the factorizations* (2.1) *and* (2.6) *satisfying conditions* (2.3) *and* (2.7), *respectively. A necessary and sufficient condition for the equality*

(2.12) $$\det Q = \det P$$

to be satisfied, up to a constant factor of modulus one, is

(2.13) $$(P_1, D)_R = I_N.$$

Proof. The necessity of condition (2.13) follows by reasoning analogous to that used in the proof of Lemma 2.2.

To prove sufficiency of condition (2.13) for (2.12) to hold, we note that

$$AD^* = \bar{\chi}PC^*D^* = \bar{\chi}C_1^*P_1D^* = \bar{\chi}QH^*.$$

Since $P = QS$, it follows that $SC^*D^* = QH^*$. Now $(P,C)_R = I_M$ implies $(S,C)_R = I_M$ and hence, by an application of Theorem 1.1, $SC^* = C_2^*S_2$ for some inner function S_2 in $H^\infty(B(N,N))$ and C_2 in $H^\infty(B(M,N))$ for which $(S_2,C_2)_L = I_N$. As $P_{H^2(M)}\tilde{\chi}H^*\xi = 0$ for all $\xi \in L$, we have $P_{H^2(M)}C_2^*(\tilde{\chi}S_2D^*\xi) = 0$, which in turn implies that $P_{H^2(M)}C_2^*P_{H^2(N)}\tilde{\chi}S_2D^*\xi = 0$. But $P_{H^2(N)}\tilde{\chi}S_2D^*\xi = H_{\tilde{\chi}S_2D^*}\xi$ belongs to $H(S_2)$. It follows from Theorem 2.6 in [3] that $P_{H^2(M)}C_2^*f = 0$ for some nonzero f in $H(S_2)$ if and only if $(C_2,S_2)_L \neq I_N$. Thus, since $(C_2,S_2)_L = I_N$, we have $H_{\tilde{\chi}S_2D^*}\xi = 0$ for all ξ in L. This, together with $(S_2,D)_R = I_N$, implies that Range $H_{\tilde{\chi}S_2D^*} = \{0\}$ and hence the triviality of S_2, i.e. S_2 and hence also S, which is quasiequivalent to it, are constant unitary operators. The primeness condition $(S_2,D)_R = I_N$ itself follows easily from (2.13) and the fact that $P = QS$. Since S is trivial, we have (2.12) satisfied up to the multiplicative constant $\det S$.

As was the case with Lemma 2.2, the above lemma can be sharpened as follows.

LEMMA 2.6. *Let A be strictly noncyclic in $L^\infty(B(N,M))$, having the factorizations* (2.1) *that satisfy conditions* (2.3), *and assume that* Range H_A *is closed. Let D belong to $H^\infty(B(N,L))$, then a necessary and sufficient condition for the equality*

$$\text{(2.14)} \qquad \text{Range } H_{AD^*} = \text{Range } H_A$$

to hold is

$$\text{(2.15)} \qquad [P_1,D]_R = I_N.$$

PROOF. We begin by proving the necessity of (2.15). By Lemma 2.5 for $\overline{\text{Range } H_{AD^*}} = \overline{\text{Range } H_A}$ it is necessary that $(P_1,D)_R = I_N$. Thus $P_1D^* = D_2^*P_2$ for an inner function P_2 in $H^\infty(B(L,L))$ and D_2 in $H^\infty(B(N,L))$ satisfying $(P_2,D_2)_L = I_L$. Since $AD^* = \tilde{\chi}PC^*D^* = \tilde{\chi}C_1^*P_1D^* = \tilde{\chi}C_1^*D_2^*P_2$, it follows that a necessary condition for Range $H_{AD^*} = H(P)$ is that $[P_2,D_2C_1]_L = I_L$. Thus it is also necessary that $[P_2,D]_L = I_L$ be satisfied which, by Theorem 1.2, is equivalent to (2.15).

Conversely, we assume (2.15) holds. Thus Range $H_{\tilde{\chi}P_1D^*} = H(P_1)$. Clearly also Range $H_{\tilde{\chi}P_1} = H(P_1)$. Now for $f \in H^2(L)$

$$H_{AD^*}f = P_{H^2(M)}AD^*(Jf) = P_{H^2(M)}\tilde{\chi}C_1^*P_1D^*(Jf) = P_{H^2(M)}C_1^*P_{H^2(N)}\tilde{\chi}P_1D^*(Jf).$$

Hence Range $H_{AD^*} = \{P_{H^2(M)}C_1^*g \mid g \in H(P_1)\}$ or

$$\begin{aligned}\text{Range } H_{AD^*} &= \{P_{H^2(M)}C_1^*P_{H^2(N)}\tilde{\chi}P_1(Jf) \mid f \in H^2(L)\}\\ &= \{P_{H^2(M)}\tilde{\chi}C_1^*P_1(Jf) \mid f \in H^2(L)\} = \text{Range } H_A = H(P).\end{aligned}$$

Finally we combine the results of the previous lemmas to give the following theorem.

THEOREM 2.7. *Let* $A \in L^\infty(B(N,M))$ *and* $B \in L^\infty(B(L,N))$ *be strictly non-cyclic, having the factorizations* (2.1) *and* (2.2) *that satisfy the primeness conditions* (2.3) *and* (2.4), *respectively. Let* Γ *be defined by* χAB *have the factorizations* (2.6) *satisfying the primeness conditions* (2.7).

(a) *A necessary and sufficient condition for*

$$\det Q = (\det P) \cdot (\det R) \tag{2.16}$$

to hold is that

$$(C,R)_L = I_N \text{ and } (P_1,D_1)_R = I_N \tag{2.17}$$

are satisfied.

(b) *Assume* H_A *and* H_B *have closed range, then* H_Γ *has closed range and* (2.16) *holds if and only if*

$$[C,R]_L = I_N \text{ and } [P_1,D_1]_R = I_N \tag{2.18}$$

are satisfied.

PROOF. (a) The necessity of conditions (2.17) for (2.16) to hold follows by the same arguments as in Lemma 2.2. So we assume (2.17) to hold and consider $\tilde{\chi}PC^*R = \tilde{\chi}C_1^*P_1R$. As $(R,C)_L = I_N$, we have $(P_1R,C_1)_L = I_N$ and thus the range closure of $H_{\tilde{\chi}PC^*R}$ is $H(Q')$ for some inner function Q' in $H^\infty(B(M,M))$, satisfying $\det Q' = (\det P_1)\cdot(\det R) = (\det P)\cdot(\det R)$. Next we consider $\Gamma = \chi AB = (\tilde{\chi}C_1^*P_1R)D^*$. By Lemma (2.5) we will have (2.16) satisfied if and only if $(P_1R,D)_R = I_N$. Now $P_1RD^* = P_1D_1^*R_1$ and by considering the range of the Hankel operators induced by these functions and using Lemma 2.2 we find that $(P_1R,D)_R = I_N$ is equivalent to $(P_1,D_1)_R = I_N$ and $(D_1,R_1)_L = I_L$, and hence sufficiency has been proved.

(b) By part (a) the weaker conditions (2.17) are necessary for (2.16) to hold. From (2.17) it follows that $C^*R = R_2C_2^*$ for some inner function $R_2 \in H^\infty(B(M,M))$ and $C_2 \in H^\infty(B(M,N))$ satisfying $(R_2,C_2)_R = I_M$. So, as $\Gamma = \tilde{\chi}PC^*RD^* = \tilde{\chi}PR_2C_2^*D^*$, for H_Γ to have closed range it is necessary that

$$[PR_2, DC_2]_R = I_M. \tag{2.19}$$

For (2.19) to hold it is necessary that

$$[R_2, C_2]_R = I_M. \tag{2.20}$$

But, by Theorem 1.2, $[R_2, C_2]_R = I_M$ if and only if $[R, C]_L = I_N$. The necessity of $[P_1, D_1]_R = I_N$ is proved analogously using the representation $\Gamma = \tilde{\chi} C_1^* P_1 D_1^* R_1$.

Conversely, let us assume the conditions in (2.18) are satisfied. Thus $\overline{\text{Range } H_\Gamma} = H(Q)$ and (2.16) is satisfied by part (a). Hence we have to prove only that Range H_Γ is closed. By Lemma 2.3, $H_{\tilde{\chi}PC^*R}$ has closed range. Now $H_{\tilde{\chi}PC^*R} = H_{\tilde{\chi}C_1^*P_1R}$ and hence

$$[P_1 R, C_1]_L = I_N \tag{2.21}$$

holds. By Lemma 2.6, H_Γ has closed range if and only if $[P_1 R, D]_R = I_N$. The range of $H_{\tilde{\chi}P_1D_1^*}$ is closed by the assumption (2.18), thus, since $\tilde{\chi} P_1 R D^* = \tilde{\chi} P_1 D_1^* R_1$, the assumption $[R_1, D_1]_L = I_L$ implies, by an application of Lemma 2.3, the range closure of $H_{\tilde{\chi}P_1RD^*}$. Hence $[P_1 R, D]_R = I_N$ and Range H_Γ is closed.

3. On inner functions and invariant subspaces

We devote this section to some results concerning inner functions and invariant subspaces that will be needed in the next section. This generalizes some results obtained in [7].

Let $R \in H^\infty(B(M,M))$ be an inner function, M being again a finite dimensional Hilbert space. Define a map $\tau_R : L^2(M) \to L^2(M)$ by

$$\tau_R f = \tilde{\chi} \tilde{R} J f. \tag{3.1}$$

It has been proved in [2] that τ_R is a unitary map that satisfies

$$\tau_R(H(R)) = H(\tilde{R}),$$

$$\tau_R(RH^2(M)) = L^2(M) \ominus H^2(M),$$

and

$$\tau_R(L^2(M) \ominus H^2(M)) = \tilde{R} H^2(M).$$

Therefore $H_{\tilde{\chi}\tilde{R}} = P_{H^2(M)} \tau_R \mid H^2(M)$ is a partial isometry, with $H(R)$ as initial space and $H(\tilde{R})$ as final space. From general properties of partial isometries it follows that

$$P_{H(R)} = H_{\tilde{\chi}\tilde{R}}^* H_{\tilde{\chi}\tilde{R}} = H_{\tilde{\chi}R} H_{\tilde{\chi}\tilde{R}}. \tag{3.2}$$

As before $P_{H(R)}$ denotes the orthogonal projection of $H^2(M)$ onto $H(R)$.

LEMMA 3.1. *Let P and R be inner functions in $H^\infty(B(M,M))$. If $(P,R)_L = I_M$, then $P_{H(R)}\{PH^2(M)\}$ is dense in $H(R)$.*

PROOF. By the representation (3.2) of the orthogonal projection on $H(R)$ we have

$$P_{H(R)}Pf = H_{\bar{\chi}R}H_{\bar{\chi}\tilde{R}}Pf = H_{\bar{\chi}R}P_{H^2(M)}\bar{\chi}\tilde{R}JPf = H_{\bar{\chi}R}P_{H^2(M)}\bar{\chi}\tilde{R}\tilde{P}^*Jf.$$

Now $(P,R)_L = I_M$ implies $(\tilde{P},\tilde{R})_R = I_M$ and hence, by Theorem 1.1, the range of $H_{\bar{\chi}\tilde{R}\tilde{P}^*}$ is dense in $H(\tilde{R})$. Moreover, since $\ker H_{\bar{\chi}R} = \tilde{R}H^2(M)$, it follows that $H_{\bar{\chi}R}(H(\tilde{R})) = H(R)$. This proves the lemma.

Using Theorem 1.2 concerning range closure of Hankel operators, we can strengthen the previous lemma as follows.

LEMMA 3.2. *Let P and R be inner functions in $H^\infty(B(M,M))$. If $[P,R]_L = I_M$ then $P_{H(R)}\{PH^2(M)\} = H(R)$.*

PROOF. From the proof of the previous lemma we have

$$P_{H(R)}\{PH^2(M)\} = H_{\bar{\chi}R}\,\{\text{Range } H_{\bar{\chi}\tilde{R}\tilde{P}^*}\}.$$

By Theorem 1.2, $[P,R]_L = I_M$ implies that $\text{Range}\, H_{\bar{\chi}\tilde{R}\tilde{P}^*} = H(\tilde{R})$. Since $\ker H_{\bar{\chi}R} = \tilde{R}H^2(M)$ and Range $H_{\bar{\chi}R} = H(R)$, the result follows.

The next theorem is quoted from [6].

THEOREM 3.3. *Let P, R and Q be inner functions in $H^\infty(B(M,M))$ for which*

$$QH^2(M) = PH^2(M) \cap RH^2(M). \tag{3.3}$$

Then there exist inner functions P_1 and R_1 such that

(i) *The factorizations*

$$Q = PR_1 = RP_1 \tag{3.4}$$

hold.

(ii) $\det Q \mid (\det P)\cdot(\det R)$ *with*

$$\det Q = (\det P)\cdot(\det R), \tag{3.5}$$

the equality up to a constant factor of modulus one, if and only if $(P,R)_L = I_M$.

(iii) *$(P,R)_L = I_M$ if and only if $(P_1,R_1)_R = I_M$.*

Equivalently, if $H(Q) = H(P) \vee H(R)$ then (3.5) is satisfied if and only if $(P,R)_L = I_M$. The condition $(P,R)_L = I_M$ is in turn equivalent to $H(P) \cap H(R) = \{0\}$. Thus equality (3.5) is the multiplicative analog of the fact that, given two finite dimensional subspaces M and N of a linear space, $\dim(M+N) = \dim M + \dim N$ if and only if $M \cap N = \{0\}$.

Replacing the left primeness of P and R by strong left primeness yields the following strengthening of statement (iii) in Theorem 3.3.

LEMMA 3.4. *Let P, R and Q be as in Theorem 3.3, then $[P,R]_L = I_M$ if and only if $[P_1,R_1]_R = I_M$.*

PROOF. Assume $[P,R]_L = I_M$. From the factorizations (3.4) it follows that $\bar{\chi}R^*P = \bar{\chi}P_1R_1^*$. By Theorem 1.2 the Hankel operator $H_{\bar{\chi}R^*P}$ has closed range and hence also its adjoint $H_{\bar{\chi}\tilde{R}_1\tilde{P}_1^*}$ has closed range. Thus $(\tilde{R}_1, P_1]_L = I_M$ or $[R_1,P_1]_R = I_M$. The converse follows by symmetry.

As a corollary we get the following theorem.

THEOREM 3.5. *Let P and R be inner functions. The left invariant subspaces $H(P)$ and $H(R)$ have a nonzero angle if and only if $[P,R]_L = I_M$.*

PROOF. The sum of two subspaces M_1 and M_2 of a Banach space satisfying $M_1 \cap M_2 = \{0\}$ is closed if and only if for some $d > 0$

$$\inf(\|x_1 - x_2\| \mid x_i \in M_i,\ \|x_i\| = 1\} \geqq d.$$

For this the reader is referred to [12]. In a Hilbert space this is equivalent to

$$\sup\{|(x_1,x_2)| \mid x_i \in M_i, \|x_i\| = 1\} < 1,$$

which has the interpretation that M_1 and M_2 have a positive angle.

Now assume $[P,R]_L = I_M$. Already the weaker condition $(P,R)_L = I_M$ implies that $H(P) \cap H(R) = \{0\}$. Thus it suffices to show

$$H(Q) = H(P) + H(R). \tag{3.6}$$

By Lemma 3.4, $[P,R]_L = I_M$ if and only if $[P_1,R_1]_R = I_M$. By the matrix version of the Carleson corona theorem [2], there exist Φ and Ψ in $H^\infty(B(M,M))$ for which $\Phi P_1 + \Psi R_1 = I_M$, and hence, using the factorizations (3.4), we have

$$Q^* = \Phi P_1 Q^* + \Psi R_1 Q^* = \Phi R^* + \Psi P^*.$$

Going to adjoints we have

$$\bar{\chi}Q = \bar{\chi}(R\Phi^* + P\Psi^*),$$

and proceed to consider the corresponding Hankel operators. $H^*_{\bar{\chi}Q}$ has closed range equal to $H(Q)$, whereas Range $H_{\bar{\chi}R\Phi^*} \subset H(R)$ and Range $H_{\bar{\chi}P\Psi^*} \subset H(P)$. So we have

$$H(Q) = \text{Range}\, H_{\bar{\chi}(R\Phi^* + P\Psi^*)} \subset \text{Range}\, H_{\bar{\chi}R\Phi^*} + \text{Range}\, H_{\bar{\chi}P\Psi^*} \subset$$
$$H(R) + H(P) \subset H(R) \vee H(P) = H(Q).$$

Hence we must have equality all the way and in particular (3.6) holds.

Next we show the necessity of $[P,R]_L = I_M$. Assume $[P,R]_L \neq I_M$. The most obvious violation of $[P,R]_L = I_M$ is the existence of a vector $\eta \neq 0$ and a point λ in the open unit disc for which

$$P(\lambda)^*\eta = R(\lambda)^*\eta = 0. \tag{3.7}$$

But $P(\lambda)^*\eta = 0$ implies $(1-\bar{\lambda}\chi)^{-1}\eta \in H(P)$, and hence (3.7) implies $H(P) \cap H(R) \neq \{0\}$.

In general (3.7) does not hold and we will resort to an approximation argument. If $[P,R]_L \neq I_M$, there exists a sequence λ_n of points in the open unit disc and a sequence of unit vectors $\eta_n \in M$ for which

$$\lim \| P(\lambda_n)^*\eta_n \| = \lim \| R(\lambda_n)^*\eta_n \| = 0.$$

We will show the existence of a sequence $F_n \in H(P)$ and a sequence $F'_n \in H(R)$ for which $\lim \| F_n \| = \lim \| F'_n \| = 1$ and also $\lim\,(F_n, F'_n) = 1$. This implies that $H(P)$ and $H(R)$ have zero angle.

Consider the normalized eigenfunctions of the left shift in $H^2(M)$ given by

$$H_n = (1-|\lambda_n|^2)^{1/2}(1-\bar{\lambda}_n\chi)^{-1}\eta_n$$

and take their decomposition with respect to the two direct sum representations of $H^2(M)$ induced by the left invariant subspaces $H(P)$ and $H(R)$. Thus $H_n = F_n + G_n = F'_n + G'_n$ with $F_n \in H(P)$, $G_n \in PH^2(M)$, $F'_n \in H(R)$ and $G'_n \in RH^2(M)$. A simple computation [3] yields

$$F_n = (1-|\lambda_n|^2)^{1/2}(1-\bar{\lambda}_n\chi)^{-1}(I_M - PP(\lambda_n)^*)\eta_n$$

and similarly for F'_n. Since P is inner, we have $\| G_n \| = \| P(\lambda_n)^*\eta_n \|$ and hence $\lim \| G_n \| = 0$ and $\lim \| F_n \| = 1$.Similarly, $\lim \| F'_n \| = 1$ and $\lim \| G'_n \| = 0$. Now

$$1 = (H_n, H_n) = (F_n + G_n, F'_n + G'_n) = (F_n, F'_n) + (F_n, G'_n) + (G_n, F'_n) + (G_n, G'_n),$$

and, as the last three terms obviously tend to zero, we have $\lim\,(F_n, F'_n) = 1$. This completes the proof.

4. Hankel operators induced by sums

Let A and B be two strictly noncyclic functions in $L^\infty(B(N,M))$ which, by Theorem 1.1, have the following prime factorizations on the unit circle:

$$A = \tilde{\chi} P C^* = \tilde{\chi} C_1^* P_1 \tag{4.1}$$

and

$$B = \tilde{\chi} R D^* = \tilde{\chi} D_1^* R_1. \tag{4.2}$$

Here P and R are inner functions in $H^\infty(B(M,M))$, P_1 and R_1 inner functions in $H^\infty(B(N,N))$, and C, C_1, D and D_1 in $H^\infty(B(M,N))$. Since the factorizations (4.1) and (4.2) are assumed prime, we have

$$(P,C)_R = I_M, \qquad (P_1,C_1)_L = I_N \tag{4.3}$$

and

$$(R,D)_R = I_M, \qquad (R_1,D_1)_L = I_N. \tag{4.4}$$

Clearly $A + B$ has also a meromorphic extension of bounded type to D_e and hence, by Theorem 1.1, it has the following prime factorizations on the unit circle:

$$A + B = \tilde{\chi} S H^* = \tilde{\chi} H_1^* S_1, \tag{4.5}$$

satisfying

$$(S,H)_R = I_M \quad \text{and} \quad (S_1,H_1)_L = I_N. \tag{4.6}$$

Here S and S_1 are inner functions in $H^\infty(B(M,M))$ and $H^\infty(R(N,N))$, respectively, and H, $H_1 \in H^\infty(B(M,N))$. Let $Q \in H^\infty(B(M,M))$ be an inner function for which

$$QH^2 = PH^2(M) \cap RH^2(M). \tag{4.7}$$

The existence of such a Q follows from the fact that $PH^2(M) \cap RH^2(M)$ is an invariant subspace of full range. By Theorem 10 in [9], there exist inner functions P' and R' in $H^\infty(B(M,M))$ for which

$$Q = PR' = RP' \tag{4.8}$$

holds. The following relations are immediate:

$$\text{Range}\, H_{A+B} = \text{Range}(H_A + H_B) \subset \text{Range}\, H_A \vee \text{Range}\, H_B$$

$$= H(P) \vee H(R) = H(Q).$$

Therefore, since $\overline{\text{Range}\, H_{A+B}} = H(S)$, we have the inclusion $H(S) \subset H(Q)$ which implies the existence of an inner function $W \in H^\infty(B(M,M))$ for which $Q = SW$. From this factorization it follows that $\det S \mid \det Q$. Now we have always $\det Q \mid (\det P) \cdot (\det R)$ with the equality

(4.9) $$\det Q = (\det P)\cdot(\det R)$$

holding, up to a constant factor of modulus one, if and only if

(4.11) $$(P, R)_L = I_M.$$

Thus (4.10) is a necessary condition for the equality

(4.11) $$\det S = (\det P)\cdot(\det R)$$

to be satisfied. Since $H(S) \subset H(Q)$, the equality (4.11) implies the equality

(4.12) $$H(S) = H(Q).$$

Condition (4.10) by itself is not sufficient for (4.12) to hold. The exact statement is given by the next theorem.

THEOREM 4.1. *Let A and B be strictly noncyclic with factorizations* (4.1) *and* (4.2) *satisfying the primeness conditions* (4.3) *and* (4.4). *Let Q be an inner function satisfying* (4.7). *A necessary and sufficient condition for the equality*

(4.13) $$\overline{\text{Range } H_{A+B}} = H(Q)$$

to hold together with relation (4.9) *is*

(4.14) $$(P, R)_L = I_M \quad \textit{and} \quad (P_1, R_1)_R = I_N.$$

PROOF. We saw already the necessity of $(P, R)_L = I_M$. Since, by Theorem 1.1, the equalities

$$\det S = \det S_1,\ \det P = \det P_1 \text{ and } \det R = \det R_1$$

hold, by considering the functions $\tilde{A}$, $\tilde{B}$ and $\tilde{A} + \tilde{B}$, it follows by the same reasoning that $(\tilde{P}_1, \tilde{R}_1)_L = I_N$ is necessary for (4.11) to hold. But $(\tilde{P}_1, \tilde{R}_1)_L = I_N$ is equivalent to $(P_1, R_1)_R = I_N$ and the necessity of (4.14) is proved.

Next assume conditions (4.14) hold. In particular $(\tilde{P}_1, \tilde{R}_1)_L = I_N$. Now $\ker H_B = \tilde{R}_1 H^2(N)$. Let us look at H_{A+B} restricted to $\ker B$. For f in $H^2(N)$ we have

$$H_{A+B}\tilde{R}_1 f = (H_A + H_B)\tilde{R}_1 f = H_A \tilde{R}_1 f.$$

But

$$H_A \tilde{R} f = P_{H^2(M)} A J \tilde{R}_1 f = P_{H^2(M)} \chi C_1^* P_1 R_1^* J f = H_{\chi C P_1 R_1^*} f,$$

and hence we have

$$\text{Range } H_{\chi C_1^* P_1 R_1^*} = H_A\{\tilde{R}_1 H^2(N)\}.$$

Now, by Lemma (2.5) the condition $(P_1,R_1)_R = I_N$ implies that

$$H(P) = \overline{\text{Range } H_{\bar{\chi}C_1^* P_1 R_1^*}}$$

To sum up, we proved $H(P) \subset \overline{\text{Range } H_{A+B}}$ and analogously $H(R) \subset \overline{\text{Range } H_{A+B}}$, and hence

$$H(P) \vee H(R) \subset \overline{\text{Range } H_{A+B}}.$$

Since the inverse inclusion holds always, we must have equality. The condition $(P,R)_L = I_M$ implies now, by Theorem 3.3, that (4.9) holds.

The corresponding result about range closure of sums of Hankel operators of closed range is given by the next theorem.

THEOREM 4.2. *Let A and B be strictly noncyclic in $L^\infty(B(N,M))$, with factorizations* (4.1) *and* (4.2) *satisfying* (4.3) *and* (4.4), *and assume H_A and H_B have closed ranges. Then* Range $H_{A+B} = H(Q)$ *for an inner function Q satisfying* (4.9) *if and only if the following strong primeness conditions hold*:

$$[P,R]_L = I_M \quad \textit{and} \quad [P_1,R_1]_R = I_N. \tag{4.15}$$

PROOF. We begin proving the necessity of conditions (4.15) for (4.9) to hold. By the previous theorem conditions (4.14) are necessary, and will be assumed to hold. This implies that Range $H_{A+B} = H(Q)$. Now Q has the two factorizations given by (4.8), and therefore from

$$A + B = \bar{\chi}\{PC^* + RD^*\} = \bar{\chi}QH^*$$

it follows that

$$H = CR' + DP'. \tag{4.16}$$

For H_{A+B} to have closed range it is necessary that

$$[Q, CR' + DP']_R = I_M \tag{4.17}$$

holds. Keeping in mind the factorizations (4.8), this implies the necessity of $[R',P']_R = I_M$, which, by Lemma 3.4, is equivalent to $[P,R]_L = I_M$. The necessity of $[P_1,R_1]_R = I_N$ follows by the same reasoning by duality.

To prove sufficiency, assume conditions (4.15) to hold. By assumption Range $H_A = H(P)$ and Range $H_B = H(R)$. By Theorem 3.5 the strong primeness condition $[P,R]_L = I_M$ implies that the angle between $H(P)$ and $H(R)$ is positive, and hence $H(P) + H(R)$ is a closed left invariant subspace. Since

$$H(Q) = H(P) \vee H(R)$$

it follows that actually

$$H(Q) = H(P) + H(R).$$

Thus it suffices to show that $H(P)$ and $H(R)$ are included in Range H_{A+B}. Since $\ker H_B = \tilde{R}_1 H^2(N)$, we have $H_{A+B}\{\ker H_B\} = H_A\{\ker H_B\} = H_A\{\tilde{R}_1 H^2(N)\}$. As in the proof of Theorem 4.1, we have

$$H_A\{\tilde{R}_1 H^2(N)\} = \text{Range } H_{AR_1^{\sim}}.$$

Applying Lemma 2.6, the condition $[P_1, R_1]_R = I_N$ implies

$$\text{Range } H_{AR_1^{\sim}} = \text{Range } H_A = H(P).$$

Similarly $H(R) \subset \text{Range } H_{A+B}$ is proved and with it the proof is complete.

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